On the Cohen-Macaulay Property of Monomial Ideals in Conical Algebras

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ABSTRACT

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(Under the direction of James N. Damon)

Cohen-Macaulay rings are an important class of rings in commutative algebra. A ring R is Cohen-Macaulay if depth R = dim R (viewed as a module). This equality of these two invariants gives rise to many important algebraic and geometric results. In this thesis, we will summarize some of these important results. We will also give different methods for calculating the depth of a module and apply them to a special class of rings, the conical algebras. We will also discuss more recent results showing when certain quotients of these conical algebras are Cohen-Macaulay.

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CHAPTER 1

Cohen-Macaulay Rings

1.1. Introduction

In geometry, a question that naturally arises is the following: given a system of polynomial equations $f_i(z) = 0$, $i = 1, 2, \ldots, k$ on $\mathbb{C}^n$, what is the structure of the set of solutions? One would like that with each additional equation, the dimension of the solution set would decrease by one, so that the solution set of the entire system, X, will have codimension k. In general, this is not true. However, the condition that there is an algebraic number, k, which equals the geometric codimension is a desirable property. An important concept in commutative algebra (and algebraic geometry) was introduced to exactly capture this idea. It is the notion that the associated coordinate ring of X has its depth equal to its dimension.

Macaulay [**8**] introduced ideas related to this notion in the early 1900's in his study of rings with determinantal relations. Cohen [**3**] further refined this in the case of complete local rings. Eventually this led to the notion of being Cohen-Macaulay, whose validity and application (which now generally applies to modules) has been studied by many workers.

In this thesis, we will survey the results concerning the Cohen-Macaulay property.

In Chapter 1, we begin by describing the basic properties of depth needed for Cohen-Macaulay properties, and give several fundamental motivating examples. We also describe several basic consequences for Cohen-Macaulay rings and modules. Second, we will explain in Chapter 2 several of the homological-based methods which one can use to calculate depth. These are what makes the notion of depth so useful in its own right.

In Chapter 3, we will consider how these methods apply to the important class of conical algebras and quotients of these algebras by monomial ideals. Conical algebras are the coordinate rings of affine toric varieties (which have natural torus actions) and these are the geometric building blocks for global toroidal varieties. Monomial ideals define torus invariant subvarieties. We will survey both the earlier results of Hochster [**10**] and more recent results of Stanley-Reisner ([**14**] and [**17**]), Bayer-Peeva-Sturmfels [**1**] and Miller [**13**]. We will conclude by discussing a possible approach to a remaining key unresolved question.

1.2. Definitions

R shall be a Noetherian ring and M a finitely generated R-module, unless otherwise noted. An x $\in$ R is **M-regular** if it is a nonzerodivisor of M, i.e., the map M $\xrightarrow{x}$ M is injective. A sequence $x_1, x_2, \ldots, x_n \in$ R is called a **M-regular sequence** (or an M-**sequence**) if

(1) $(x_1, \ldots, x_n)$M $\neq$ M, and

(2) For i=1,...,n, x_i is a nonzerodivisor on M/$(x_1, \ldots, x_{i-1})$M.

Definition 1.2.1. *If I is an ideal of R and M is a finitely generated R-module, then the* ***depth*** *of I on M, denoted* ***depth(I,M)****, is the length of a maximal M-sequence in I.*

If IM=M, we define depth(I,M) = ∞.

DEFINITION 1.2.2. *We shall refer to (R,$\mathfrak{m}$) as a* ***graded ring*** *if R is a positively graded algebra over* k*, a algebraically closed field of characteristic 0, and* $\mathfrak{m}$ *is the unique maximal homogeneous ideal.*

For all results we state for local rings, (R,$\mathfrak{m}$), there are analogous results for graded rings (R,$\mathfrak{m}$). Also, some of the results to be described are, in fact, valid over algebraically closed fields of positive characteristic, but we shall not consider them here. The depth of M, denoted **depth M**, will refer to depth($\mathfrak{m}$,M). Since the action of x $\in$ R depends on x modulo ann(M), we will only consider ideals that contain ann(M).
We recall the following definition for an R-module M.

DEFINITION 1.2.3. *The dimension of M,* ***dim M****, is the supremum over the lengths t of strictly descending chains*

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_t,$$

where $\mathfrak{p}_i \in$ *Supp(M).*

We will usually deal with the case where M is a finitely generated R-module, so that one has Supp M = $\{\mathfrak{p} \in \text{Spec R} |\ \mathfrak{p} \supset \text{Ann(M)}\ \}$. Then, dim M = dim (R/(Ann M)). We recall the next definition for an R-module M.

DEFINITION 1.2.4. *Let (R,$\mathfrak{m}$) be a local ring or a graded ring. An R-module M is said to be a* ***Cohen-Macaulay (C-M) module*** *if depth M = dim M. R is Cohen-Macaulay if it is a C-M module over itself. More generally, for an arbitrary Noetherian ring R, a module M is Cohen-Macaulay if* $M_{\mathfrak{m}}$ *is a C-M module over* $R_{\mathfrak{m}}$ *for all maximal ideals* $\mathfrak{m}$ *of R.*

Philosophically, we have two different measures of the extent of module M, one that is very algebraic (depth) and the other which is more geometric in nature (dimension). The Cohen-Macaulay property tells us when both these measures coincide.

1.3. Motivating Examples

We first give some examples of classes of rings to serve as a motivation for studying the Cohen-Macaulay property. The first example, regular local rings, is easily seen to be C-M. The second class, determinantal rings, were originally considered by Macaulay. His work led to the development of the theory we have described. From these classes we can generate further examples.

EXAMPLE 1.3.1. *Regular Local Rings*

If R is a regular local ring, then by definition the number of generators of the maximal ideal is equal to the dimension of R. If one has a regular system of parameters, it forms a R-sequence. Combining these two statements gives us that regular local rings are Cohen-Macaulay.

EXAMPLE 1.3.2. *Determinantal Rings*

Let k[X] be the polynomial ring in the entries of a m$\times$n matrix X of indeterminates. Let I_{r+1} be the ideal generated by all (r+1)-minors of X, $0 \leq r \leq$ rank(X)-1. Then $R_{r+1} = k[X]/ I_{r+1}$ is called a determinantal ring of dimension (m+n-r)r. Macaulay studied the case where X is the *generic* matrix, where each entry is a distinct indeterminate.

For example, let

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

The ideal generated by the 2×2 minors of X is

$$I_2 = < x_1x_5 - x_2x_4,\ x_1x_6 - x_3x_4,\ x_2x_6 - x_3x_4 >.$$

Macaulay proved that for r+1=min(m,n), the ideals I_{r+1} have no embedded primes, i.e., they are unmixed [**8**]. In [**5**], Eagon and Northcott constructed a finite free resolution of the ideals I_{r+1} and proved that the Cohen-Macaulay property of these determinantal rings is equivalent to the perfection of these ideals. Later, Eagon and Hochster proved that R_{r+1} is Cohen-Macaulay for general r, $0 \leq r \leq$ rank(X)-1 [**11**]. As a more general example, let

$$X = \begin{pmatrix} x_1 & x_2^3 & x_3 + x_4 \\ x_3x_4 & x_5 & x_4^2 \end{pmatrix}.$$

The ideal generated by the 2×2 minors of X is given by

$$I = < x_1x_5 - x_2^3x_3x_4,\ x_1x_4^2 - x_3^2x_4 - x_3x_4^2,\ x_2^3x_4^2 - x_3x_5 - x_4x_5 >$$

Then, $R=k[x_1,x_2,x_3,x_4,x_5]/I$ is C-M.

When one verifies that a module is Cohen-Macaulay, a natural question is how to construct other Cohen-Macaulay modules. The next theorem deals with the case of Cohen-Macaulay rings. Recall that for a prime ideal P, the **height** of P is the supremum of the length of chains of prime ideals descending from P. More generally, the **height** of I is the minimum of the heights of primes containing I.

THEOREM 1.3.3. *If R is a Cohen-Macaulay ring and* $x_1,x_2,\ldots,x_n$ *is an R-sequence and* $I = (x_1,x_2,\ldots,x_n)$ *has height n, the largest possible value, then R/I is Cohen-Macaulay.*

1.4. Properties of Depth

In this section we discuss several properties of the module M that relate the depth to various invariants of the module. One is an inequality that always exist between the depth and dimension of M. Another is the non-vanishing of certain Ext functors. The Auslander-Buchsbaum formula relates depth to the length of the minimal free resolution of M. Collectively, these properties give us different approaches in verifying whether a given module M has the correct depth.

LEMMA 1.4.1. *Let* $(R,\mathfrak{m})$ *be a local ring and M a finitely generated R-module. Then depth* $M \leq$ *dim M.*

PROOF. We shall use induction on the depth of M. If depth M = 0, then $\mathfrak{m}$ consists of zero-divisors. Therefore $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$. Thus dim M = 0. Now assume the statement holds for depth M < n. Let $x_1, x_2, \ldots, x_n \in$ be a maximal M-sequence. Since x_1 is a nonzerodivisor, it cannot be contained in any minimal prime of R, so dim $M/(x_1M)$ (as a $R/(x_1)$-module) < dim M. On the other hand, the depth of $M/(x_1)M$, again as a $R/(x_1)$ module, is n-1. Thus, by induction we have the inequalities

$$n - 1 = \mathrm{M}/(\mathrm{x}_1)\mathrm{M} \leq \dim \mathrm{M}/(\mathrm{x}_1)\mathrm{M} \leq \dim \mathrm{M} - 1. \tag{1.1}$$

So we get that $n \leq$ dim M. □

Next, we show that the depth can be computed homologically.

Lemma 1.4.2. *Let M be a finitely generated R-module. If $I \subset R$ and I + ann(M) $\neq R$, then the depth (I,M) is the smallest integer t such that* Ext^t*(R/I,M)$\neq$0.*

Proof. First if I + ann(M) = R, then we could write s + t = 1 for some s $\in$ I and t $\in$ ann(M). Then sM = sM + 0 = (s+t)M = M. So our hypothesis is a necessary condition to have depth (I,M) $< \infty$. Again, we shall use induction on d = depth (I,M). If d = 0, we show that Ext^0(R/I,M) = Hom(R/I,M) $\neq$ 0. Since d=0, I consists of zerodivisors. Then, there exists an associated prime $\mathfrak{p}$ of M that contains I. By definition $\mathfrak{p}$ = Ann(m) for some m $\in$ M, so there exists a monomorphism R/$\mathfrak{p}$ $\to$ M. Therefore we have a non-zero map R/I $\to$ M, so Hom(R/I,M) $\neq$ 0.

Now let d $\geq$ 1, and let x $\in$ I be a nonzerodivisor on M. We have IM/(x)M $\neq$ M/(x)M, and depth (I,M/(x)M) = d-1. Consider the long exact sequence for $\mathrm{Ext}^{\cdot}$(R/I,_) for the short exact sequence

$$0\to \mathrm{M}\xrightarrow{x}\mathrm{M}\to \mathrm{M}/(\mathrm{x})\mathrm{M}\to 0$$

Since x annihilates R/I, it annihilates each Ext^j(R/I,M). Therefore Hom(R/I,M)=0 and we get short exact sequences for all j $\geq$1

$$(1.2)\quad 0 \to \mathrm{Ext}^{j-1}(R/I, M) \to \mathrm{Ext}^{j-1}(R/I, M/(x)M) \to \mathrm{Ext}^{j}(R/I, M) \to 0.$$

By induction Ext^j(R/I,M/(x)M) = 0 for j < d-1 and $\neq$ 0 for j = d-1. Therefore it follows from (1.2) that Ext^j(R/I,M) = 0 for j < d, and $\neq$ 0 for j = d. □

In particular we get the following corollary.

Corollary 1.4.3. *Let (R,$\mathfrak{m}$) be a local ring, with k the residue field. If M is a nonzero, finitely generated R-module, then depth M = min$\{i \mid \mathrm{Ext}^{\,i}(k,\mathrm{M}) \neq 0\}$.*

Another corollary is obtained from the the long exact sequence of Ext.

Corollary 1.4.4. *If 0 $\rightarrow M' \rightarrow M \rightarrow M'' \rightarrow$ 0 is an exact sequence of non-zero finitely generated R-modules, then depth M $\geq$ min{depth M',depth M''}. Also, if we have strict inequality, then depth M' = depth M''+1.*

We next explain a third way to calculate depth. This method uses the minimal projective resolution of the module and the Auslander-Buchsbaum formula.

Definition 1.4.5. *The **projective dimension**, written **pd M**, of a R-module M is the minimum of lengths of projective resolutions of M. The **global dimension** of R is the supremum of the projective dimensions of all R-modules.*

A ring with finite global dimension is a useful tool. It is known (see [**6**] Chap 19, Sect.2) that a local ring has finite global dimension iff it is a regular local ring, which we have seen is a C-M ring. In order to exploit this fact, we present the following formula which uses a connection between projective dimension with depth.

Theorem 1.4.6. (Auslander-Buchsbaum Formula)*Let (R,$\mathfrak{m}$) be a local ring. If M is a finitely generated R-module of finite projective dimension, then*

$$pd\, M = depth\, R - depth\, M. \tag{1.3}$$

This formula is also valid for (R,$\mathfrak{m}$), a positively graded ring.

As a corollary we obtain that M is C-M iff depth R - pd M = dim M. This result allows one to calculate the depth of a module by finding its minimal free resolution. Bayer, Sturmfels and Peeva use this technique to compute the depth of of generic monomial ideals in a polynomial ring [**1**] (See Chap 3, Sec 1.2 below). Note that a polynomial ring is a positively graded ring, so we can use a graded version of (1.3).

1.5. Examples

Aside from the two examples we have discussed previously, the Cohen-Macaulay property gives us a rich selection of rings to work with. We will briefly discuss several additional classes of rings which have been shown to be Cohen-Macaulay. In this section, k will be an algebraically closed field.

EXAMPLE 1.5.1. *Conical Algebras*

DEFINITION 1.5.2. *A nonempty subset C of $\mathbb{R}^n$ is a **cone** if it is closed under linear combinations with non-negative real coefficients. For $S \subset \mathbb{R}^n$ the set*

$$\mathbb{R}_+S = \{\sum_{i=1}^{n} a_i s_i \mid a_i \in \mathbb{R}_+,\ s_i \in S\}$$

*is called the cone generated by S. C is **rational** if it is generated by elements in $\mathbb{Q}^n$. C is **positive** if C lies in the positive orthant in $\mathbb{R}^n$ with vertex at the origin.*

Let C be a positive, rational cone in $\mathbb{R}^n$. Let S be the semigroup of integer points in the cone. Let k[C] be the vector space with k-basis the monomials $\mathbf{x}^\alpha = \mathrm{x}_1^{\alpha_1} \dots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in$ S. We define multiplication by $\mathbf{x}^{\alpha_1} \cdot \mathbf{x}^{\alpha_2} = \mathbf{x}^{\alpha_1+\alpha_2}$. For m $= \mathbf{x}^\alpha$, α is called the **exponent vector of m**. k[C] is called the **conical algebra** corresponding to C. k[C] is a positively (multi-)graded local ring, where $\overline{\mathfrak{m}}$ is the ideal generated by all monomials in the interior of C. We have the following theorem.

THEOREM 1.5.3. (Hochster[**10**]) *Let C be a positive, rational cone. Then k[C] is a Cohen-Macaulay ring.*

A important step in Hochster's proof was to consider a cross section of C, which is a convex polytope and use the shellability of convex polytopes. Hochster uses the above result to show that given a diagonalizable group D, i.e., direct products D = T $\times$ H where T is a torus and H is a finite Abelian group, acting linearly on a polynomial ring R = $k[x_1,x_2,\ldots,x_n]$, the ring of invariants R^D is isomorphic to a Cohen-Macaulay ring [**10**]. In particular, when D = T, R^T is a conical algebra. For a further generalization, Hochster and Eagon showed that if G is a finite group whose order, |G|, does not divide char k, and G acts linearly on a polynomial ring R, then R^G is a Cohen-Macaulay ring [**11**].

Example 1.5.4. *Invariants of Linearly Reductive Groups*

The results of Hochster for diagonalizable groups in [**10**] were extended by Hochster and Roberts. They proved for G, a linear reductive group acting linearly on a polynomial ring R = $k[x_1,x_2,\ldots,x_n]$, the ring of invariants R^G is C-M [**12**]. Thus the result about invariants of tori or finite groups above is a special case of these reductive groups. Examples of linearly reductive groups are the classical Lie groups GL(n,$\mathbb{C}$), SL(n,$\mathbb{C}$), and O(n,$\mathbb{C}$).

1.6. Properties of C-M rings and modules

We complete the chapter by summarizing several key properties of C-M rings and modules. They illustrate how broadly applicable the theory of Cohen-Macaulay rings is. The first two properties deal with localizations and extensions.

Property 1.6.1. ([**6**],Proposition 18.8) *The Cohen-Macaulay condition is a local property; that is M is a Cohen-Macaulay R-module iff* $M_{\mathfrak{p}}$ *is Cohen-Macaulay for every prime ideal* $\mathfrak{p}$.

In general, under localization by a prime ideal $\mathfrak{p} \in$ supp(M), we have depth(I,M) $\leq$ depth($I_{\mathfrak{p}}$,$M_{\mathfrak{p}}$) for any ideal I $\subset \mathfrak{p}$. However given an ideal I, there exists a maximal ideal $\mathfrak{m}$ such that localizing with respect to $\mathfrak{m}$ gives equality [**6**]. As an example, if R is a regular ring, i.e., a ring whose localizations are all regular local rings, then R is C-M, by Example 1.3.1. We say that R is a **complete intersection** if there is regular ring S and a regular sequence x_1,x_2,...,$x_n \in$ S such that R $\cong$ S/(x_1,x_2,...,x_n). As an example, if X is a 1 $\times$ n matrix of rank 1, and I was the ideal generated by the entries, then the determinantal ring k[X]/I is a complete intersection. Indeed, complete intersections are determinantal rings of rank 1, 1 $\times$ n matrices. Further, R is **locally a complete intersection** if R_m is a complete intersection for every maximal ideal m of R. By Theorem 1.3.3 and Property 1.6.1, if R is locally a complete intersection, then R is C-M.

Property 1.6.2. ([**6**],Proposition 18.9) *R is a C-M ring iff R[x_1,x_2,...,x_n] is C-M as well.*

Indeed, the proof of the forward direction uses the fact that the variables are nonzerodivisors and the dimension increases by the number of variables. The other direction uses Property 1.6.1.

The next three properties discuss how varieties associated to C-M rings must behave geometrically ([**6**] Chap. 18, Sect. 2).

Property 1.6.3. *For a local C-M ring R, any two maximal chains of prime ideals have the same length and every associated prime is minimal.*

Geometrically, 1.6.3 says if a variety X has the property that at some point p, $\mathcal{O}_{X,p}$ is Cohen-Macaulay, then p cannot lie on two different components of different dimensions.

Property 1.6.4. (Hartshorne's Connectedness Theorem) *At a C-M point, a variety cannot be disconnected by removing a subvariety of codimension 2 or more.*

For example, a variety that looks locally like two surfaces meeting at a point in four-space cannot be C-M.

We also have the following Unmixedness Theorem for C-M rings, which was the original starting point for Macaulay. Macaulay proved the result for polynomial rings and, subsequently Cohen for regular local rings. This is the reason that the rings are given the name "Cohen-Macaulay". Eisenbud describes this result as "a pillar of algebraic geometry" [**6**]. Typically, one uses the unmixedness theorem to show that given set of polynomials generates the homogeneous coordinate ring of a given projective variety.

Theorem 1.6.5. *Let R be a ring. If I =(x_1,x_2,...,x_n) is an ideal generated by n elements such that codim I = n, then all minimal primes of I have codimension n. If R is a C-M ring, then every associated prime of I is minimal over I.*

CHAPTER 2

Depth

The main difficulty in verifying the Cohen-Macaulay property of a module is in calculating the depth. That is, the existence of M-regular sequences and their maximality is not easily established directly. We discussed, in the previous chapter, a method for calculating depth using the Ext functor (Lemma 1.4.2). In this chapter, we discuss several other methods. One method uses the Koszul complex. For our purposes, it will be more illuminating to use a different method, which uses the theory of local cohomology to calculate the depth. There are a number of ways to calculate the depth of a module using local cohomology which do not require finding a regular sequence.

2.1. Koszul Complex

We first mention the relevance of the Koszul complex in calculating depth.

Let N be an R-module and x $\in$ N. We define the Koszul complex to be

$$\mathrm{K(x)}:0 \to \mathrm{R} \to \mathrm{N} \to \wedge^2\mathrm{N} \ldots \to \wedge^i\mathrm{N} \xrightarrow{d_x} \wedge^{i+1}\mathrm{N}\ldots$$

where d_x sends an element, m, of the exterior product to x $\wedge$ m. When N is a free module of rank r, let e_i, i=1,. . .,r be a basis for N. For x $= \sum x_i e_i \in$ N, we will write $\mathrm{K}(x_1,x_2,\ldots,x_r)$ instead of K(x). In this case, let m $= \sum_\sigma m_\sigma e_\sigma$, where $\sigma = \{i_1,\ldots,i_t\}$ is an increasing subsequence of $\{1,\ldots,r\}$ and $e_\sigma = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_t}$. Then

$$d_x(m) = \sum_\sigma (x_i \cdot m_\sigma)\ e_i \wedge e_\sigma.$$

We then have the following theorem ([**6**], Theorem 17.4).

THEOREM 2.1.1. *Let M be a finitely generated free R-module. If*

$$H^j(M \otimes K(x_1,x_2,\ldots,x_r)) = 0 \text{ for } j < d$$

while

$$H^d(M \otimes K(x_1,x_2,\ldots,x_r)) \neq 0,$$

then every maximal M-sequence in $I = (x_1,x_2,\ldots,x_r) \subset R$ *has length d.*

It is a simple exercise to show that $\mathrm{H}^t(\mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)) = \mathrm{R}/(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)$. Therefore if $\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t$ is an M-sequence, then $\mathrm{H}^t(\mathrm{M} \otimes \mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)) = \mathrm{M}/(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)\mathrm{M}$.

If $(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)$ is a regular sequence in R, then $\mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)$ is a free resolution of $\mathrm{R}/(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)$. If N is a free finitely generated R-module, then the Koszul complex is isomorphic to its own dual. Thus, we obtain

$$\mathrm{Hom}(\mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t), \mathrm{M}) \cong \mathrm{M} \otimes \mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t)$$

as complexes.

The homology of $\mathrm{Hom}(\mathrm{K}(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t), \mathrm{M})$ is $\mathrm{Ext}^\bullet(\mathrm{R}/(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_t), \mathrm{M})$. Therefore Theorem 2.1.1 coincides with Lemma 1.4.2. To illustrate this connection with an example, we shall show that regular local rings have finite global dimension.

DEFINITION 2.1.2. *The* ***projective dimension of a module M****, written pd M, is the minimum of the lengths of projective resolutions of M.*

DEFINITION 2.1.3. *The* ***global dimension*** *of a ring R, written gl dim R, is the supremum of projective dimensions for all finitely generated R-modules M.*

We need the following lemma to obtain our result about the global dimension of regular local rings ([**6**] Corollary 19.5).

LEMMA 2.1.4. *Let (R,$\mathfrak{m}$) be a local ring. Let k be the residue class field and M be a finitely generated nonzero R-module. Then pd M is the smallest integer i $\geq$ 0 for which Tor_{i+1}(k,M) = 0.*

PROOF. $\mathrm{Tor}_{i+1}(k,\mathrm{M})$ is defined as the (i+1) homology module of the tensor product of k with a resolution of M. Let

$$\mathcal{F}\colon\ 0\to \mathrm{F}_t \to \mathrm{F}_{n-1}\ldots\to \mathrm{F}_0 \to \mathrm{M}$$

be a resolution of length t. Let i be the smallest integer that $\mathrm{Tor}_{i+1}(k,\mathrm{M}) = 0$. Clearly, we have t $\geq$ pd M $\geq$ i. If the complex above is minimal, then the differentials of $k \otimes \mathcal{F}$ are 0. Thus, $\mathrm{Tor}_{i+1}(k,\mathrm{M}) = k \otimes \mathrm{F}_{i+1}$. This is 0 iff $\mathrm{F}_{i+1} = 0$. Thus, F_j = for j $\geq$ i+1. Therefore pd M = i. □

If R is a regular local ring of dimension n and (x_1,x_2,...,x_n) generates the maximal ideal of R, then we see that the Koszul complex K(x_1,x_2,...,x_n) is a minimal free resolution of length n of the residue class field k = R/(x_1,x_2,...,x_n). Thus, pd k = n. Since the Tor functors can be computed by taking a minimal free resolution of k, the lemma shows that for all finitely generated M, pd M $\leq$ pd k. Thus R has finite global dimension = n.

2.2. Local Cohomology

Local cohomology is an algebraic version of the topological notion of local cohomology. More specifically, the local cohomology of X at x $\in$ X, is $\mathrm{H}^\bullet(\mathrm{X},\mathrm{X}/\{\mathrm{x}\})$. In our case, where we are discussing conical algebras, the topological analogue will be the cell complex of a convex polytope determined by a rational cone C. The local cohomology modules will be $\mathbb{Z}^n$-graded, and we shall discuss a geometric criteria to determine the graded components of the various modules. We will then relate the local cohomology functors to the depth. The background information on local cohomology is excerpted from Chapter 3.5 of [**2**]. A somewhat easier method is given by Grothendieck.

Let (R,$\mathfrak{m}$) be a local ring or a graded ring, with a system of parameters $\{x_1, x_2, \ldots, x_n\}$. For an R-module M, let $\Gamma_{\mathfrak{m}}(M)=\{m \in M \mid \mathfrak{m}^k m = 0 \text{ for some } k \geq 0\}$. It is a fact that $\Gamma_{\mathfrak{m}}(\_)$ is a left exact functor and the corresponding right derived functors of $\Gamma_{\mathfrak{m}}(\_)$ will be called the *local cohomology functors* and denoted by $H^i_{\mathfrak{m}}(\_)$, $i \geq 0$.

We want to construct a complex whose cohomology gives us $H^{\bullet}_{\mathfrak{m}}(M)$. Let $R_{x_{i_1}x_{i_2}\ldots x_{i_j}}$ denote the localization of R with respect to the multiplicatively closed set $\{(x_{i_1}x_{i_2}\ldots x_{i_j})^n\}$. Define the *modified Čech complex*:

$$C^{\bullet}\colon 0 \to C^0 \to C^1 \to \ldots C^n \to 0, \quad \text{where } C^t = \bigoplus_{1\leq i_1<i_2<\ldots i_t \leq n} R_{x_{i_1}x_{i_2}\ldots x_{i_t}} \text{ and } C^0 = R,$$

The differentiation $d^t\colon C^t \to C^{t+1}$ is defined on each component

$$R_{x_{i_i}x_{i_2}\ldots x_{i_t}} \to R_{x_{j_1}x_{j_2}\ldots x_{j_t}x_{j_{t+1}}}$$

to be

$$d_t(r) = \begin{cases} (-1)^{k-1}\cdot i(r), & \text{if } \{i_1, i_2, \ldots, i_t\} = \{j_1, j_2, \ldots, \hat{j}_k, \ldots, j_{t+1}\} \\ 0, & \text{otherwise} \end{cases},$$

where i is the inclusion map.

The following theorem allows one to compute the local cohomology using the above complex ([**2**],Theorem 3.5.6).

THEOREM 2.2.1. *Let M be an R-module. Then*

$$H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes C^{\bullet}) \text{ for } i \geq 0.$$

The importance of local cohomology for understanding depth and dimension of a module results from a theorem by Grothendieck [**7**].

Theorem 2.2.2. (Grothendieck) *Let M be an R-module with dimension = d and depth = t. Then*

a)$H^i_{\mathfrak{m}}(M)=0$ *for* $i < t$ *and for* $i > d$.

b)$H^i_{\mathfrak{m}}(M)\neq 0$ *for* $i = t$ *and* $i = d$.

Hence, if M is a C-M module, then t = d and we see that there is only one non-zero local cohomology module, namely $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{M})$.

2.3. $\mathbf{L}^{\bullet}$ Complex

In the case of a conical algebra, there is a complex very similar to $\mathrm{C}^{\bullet}$ defined above which computes local cohomology and retains the geometric properties of the cone. Another property of this complex is that it comes with a natural multigrading. This complex is due to a similar one constructed by Goto and Watanabe [**16**]. We say that a hyperplane H **supports** C if $\mathrm{H} \cap \mathrm{C} \neq \emptyset$ and C is contained in one of the half-spaces defined by H. If H is a supporting hyperplane, $\mathrm{H} \cap \mathrm{C}$ is a **face** of C. A rational cones can be described as the intersection of a finite number of half-spaces, H_i^+, defined by supporting hyperplanes, where

$$\mathrm{H}_i^+ = \{\mathbf{v} \in \mathbb{R}^n \mid (\mathbf{v},\mathbf{a}_i) \geq 0\}, \quad \mathbf{0} \neq \mathbf{a}_i \in \mathbb{R}^n, \text{ i=1,2,\ldots,m}.$$

The set of faces can be partially ordered under inclusion. The complex we define in this section will use this partial ordering.

In what follows, $k[\mathrm{C}]$ will denote the conical algebra associated with a positive rational cone C. Let F be a face of C. We shall say that a monomial x^{α} lies on a face F if the monomial's exponent vector, α, lies on F. Note that if an ideal I is generated by all monomials which lie off of a face F, this ideal is prime. Indeed, if both x^α, x^β lie on F, then $x^\alpha x^\beta = x^{\alpha+\beta}$ must also lie on F. So the monomials lying in a face F make up a multiplicatively closed set. For a non-empty face F of C, we denote this corresponding graded prime ideal by $\mathfrak{p}_F$. We denote the localization $k[C]_{(\mathfrak{p}_F)}$ by $k[C]_F$. Note that this is the ring of fractions of k[C] with respect to the monomials which lie on F.

Let T be a hyperplane which is transversal to C, that is, T intersects every face of positive dimension. Let D = T $\cap$ C. We call D a cross-section of C. A cross-section D of C is a convex polytope. Let $\tilde{\mathcal{F}}$(D) be the face lattice of D (including the empty face) ordered by inclusion. (Note that we place the tilde over $\mathcal{F}$ to highlight the fact we have included the empty face.) The faces of C are given by C itself or by C $\cap$ H, where H is a supporting hyperplane. Since H must contain 0 for all supporting hyperplanes, there is a unique minimal non-empty face of C, namely $\{0\}$. Let $\mathcal{F}$(C) be the set of non-empty faces of C. The assignment Θ: F $\mapsto$ F $\cap$ T gives an isomorphism $\mathcal{F}(C) \cong \tilde{\mathcal{F}}(D)$ of partially ordered sets [**2**]. We shall denote by f the face of D such that f = F $\cap$ T.

The following facts about convex polytopes is taken from [**9**]. Let ϵ be an incidence function on $\tilde{\mathcal{F}}$(D). This means that ϵ satisfies the following:

1)ϵ: $\tilde{\mathcal{F}}(D) \times \tilde{\mathcal{F}}(D) \to \{0,\pm1\}$;

2)ϵ(f,g) $\neq 0$ iff g is a face of f;

3)$\epsilon(v,\emptyset) = 1$ for all vertices v;

4)if f is an i-dimensional face and h is an (i + 2)-dimensional face with f $\subset$ h, then

$$\epsilon(h,g_1)\epsilon(g_1,f)+\epsilon(h,g_2)\epsilon(g_2,f)=0$$

where g_1 and g_2 are the unique (i+1)-dimensional faces that are both faces of h and contain f. This function is equivalent to putting an orientation on D. We can equivalently define an incidence function on $\mathcal{F}$(C).

We can now define the L-complex associated with R = k[C] ([**2**],Chapter 6.2):

$$L^{\bullet}:0 \to L^0 \to L^1 \to L^2 \to \ldots \to L^n \to 0, \quad \text{where } L^t = \bigoplus_{\text{dim F}=t} R_F.$$

We note that $L^0 = R_{(0)} = R$. The differential is defined componentwise

$$\partial\colon R_F \to R_{F'}$$

to be

$$\partial(\mathrm{r}) = \begin{cases} \epsilon(\theta(F), \theta(F')) \cdot i(r), & \text{if } F \subset F' \\ 0, & \text{otherwise} \end{cases},$$

where i: $R_F \to R_{F'}$ is the natural inclusion map . One can note that the L-complex is an algebraic version of the augmented oriented chain complex of D with orientation given by ϵ. Let us note, with the following example, the connection between the $L^{\bullet}$ complex and the modified Čech complex $C^{\bullet}$ defined for regular local rings.

Let us consider the positive orthant as a cone. Therefore, a cross-section is the standard n-simplex, ie. the simplex spanned by the canonical basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$. Recall that for the Čech complex, the components of C^t are of the form $R_{x_{i_1}\cdots x_{i_t}}$. We can view these localizations as R_F, where F is the face of the n-simplex spanned by $e_{i_i},\ldots,e_{i_t}$. Thus, we can consider the construction of the $L^{\bullet}$ complex to be a generalization of $C^{\bullet}$. The following theorem, whose proof follows the same pattern as the proof for Theorem 2.2.1, states that this new complex provides an alternate way to compute the local cohomology ([**2**], Theorem 6.2.5).

THEOREM 2.3.1. *Let C be a positive rational cone. For every k[C]-module M and for all $i \geq 0$, we have $H^i_{\mathfrak{m}}(M) \cong H^i(L^\bullet \otimes M)$.*

With the L-complex, given a multidegree $\overline{m}$, one now has a nice geometric interpretation of how to compute the $\overline{m}$-graded component of ($L^\bullet \otimes$ R/I), denoted by $(L^\bullet \otimes R/I)_{\overline{m}}$, in terms of the cone C in $\mathbb{R}^n$. Localizing at the prime $\mathfrak{p}_F$ is to allow those monomials lying on F to be inverted. In other words, $x^\alpha \in R_F$ if there exists a monomial x^β lying in F such that $x^{\alpha+\beta} \in$ R. Geometrically, one can translate the cone C by -β and x^α will lie in this translated cone. Thus, R_F has a non-zero component in multidegree α iff x^α lies in the union of translations of C by monomials lying in the linear span of F. Note also that the ideal I can be thought of as the union of cones $C_i = \{v + m_i, v \in C\}$, which are translates of the original cone by each ideal generator x^{m_i}. So IR_F is the union of the translations of these *ideal cones* by monomials lying in the linear span of F. So $0 \neq x^{\overline{m}} \in R_F/IR_F$ iff $\overline{m}$ is in a translate of the original cone, but not in any of the translates of the ideal cones. It suffices to calculate depth for each multidegree $\overline{m}$.

EXAMPLE 2.3.2.

Let C be the region in $\mathbb{R}^2$ that is bounded by the positive x-axis and the line y = 4x. Let F be the face of C which lies on the x-axis and G the other face. Let I $\subset$ k[C] be the ideal generated by x^5y^6 and x^8y^2. The cone C along with the ideal cones are pictured in Fig 2.1. The ideal I is generated by all monomials in the union of the two cones with vertices at (5,6) and (8,2).

In Fig. 2.1, monomials in the cross-hatched region give rise to the local cohomology of k[C] being non-zero in dim 0. (See below.)

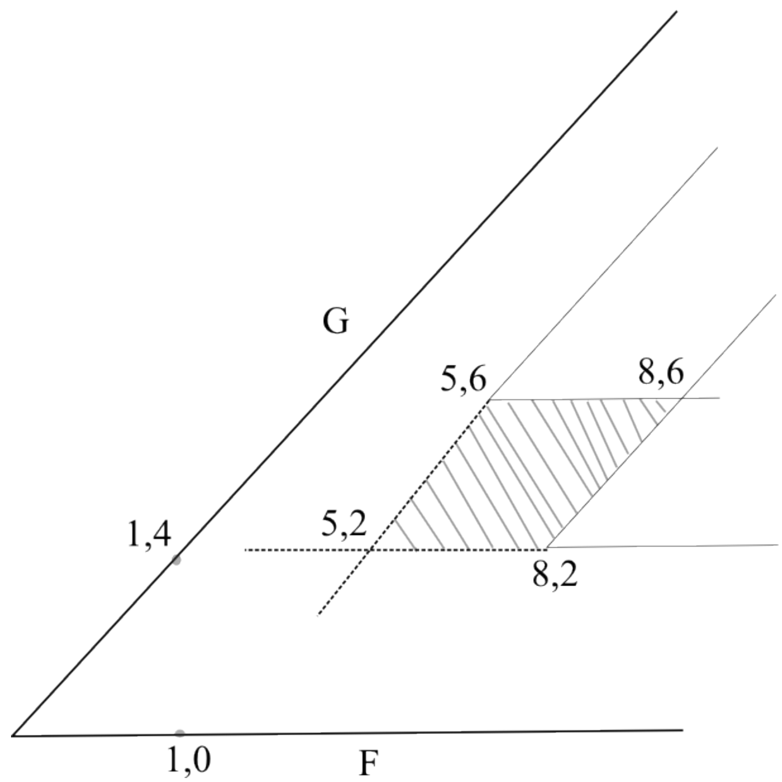

FIGURE 2.1. k[C], with I =<x^5y^6, x^8y^2 >; the existence of monomials lying in the shaded region imply k[C] is not Cohen-Macaulay

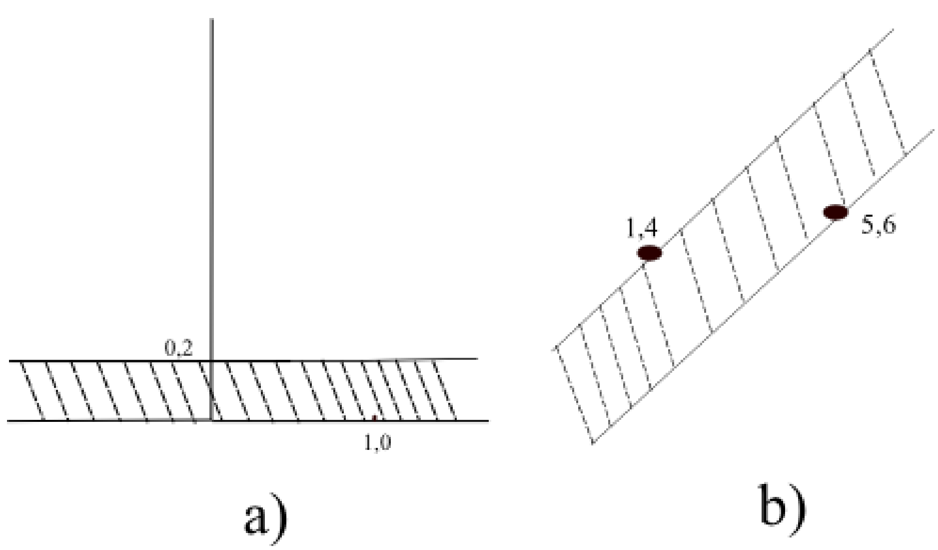

FIGURE 2.2. The local rings a)R_F/IR_F and b) R_G/IR_G

R_F is the ring generated by all monomials with a non-negative y-coordinate. Thus the cone generated by R_F is the upper-half plane. IR_F is generated by all monomials that are on the same horizontal line as a monomial in I, namely all monomials with y-coordinate ≥ 2. Therefore, R_F/IR_F is the ring generated by all monomials in the strip $\{(a,b) \mid 0 \leq b < 2\}$. This region is pictured in Fig. 2.2a above.

R_G is the union of translates of C by monomials in the linear span of G. Therefore, the localized ring is generated by monomials with integer exponent vectors (a,b) $\in$ such that b $\leq$ 4a, ie. the integer points lying in the half-space bounded above by the line y = 4x. IR_G is similarly generated by monomials with integer exponent vectors (a,b) whose dot product $(a,b) \cdot (4,-1)$ is greater than $(5,6) \cdot (4,-1) = 14$. Again, R_G/IR_G is the ring generated by all monomials in the strip $\{(a,b) \mid 0 \leq (a,b) \cdot (4,-1) < 14\}$. This is the cross-hatched region in 2.2b.

Now given these regions, we are able to calculate the local cohomology at any multi-degree. If an integer point $\overline{m}$ is in the horizontal strip depicted in Fig 2.2a, then

$$(\mathrm{L} \otimes \mathrm{R/I})_{\overline{m}} = \begin{cases} 0 \to k \to k \to 0, & \text{if } \overline{m} \in \mathrm{C} \\ 0 \to 0 \to k \to 0, & \text{if } \overline{m} \notin \mathrm{C} \end{cases}.$$

Similarly if $\overline{m}$ is in the diagonal strip in 2.2b, we see that

$$(\mathrm{L} \otimes \mathrm{R/I})_{\overline{m}} = \begin{cases} 0 \to k \to k \to 0, & \text{if } \overline{m} \in \mathrm{C} \\ 0 \to 0 \to k \to 0, & \text{if } \overline{m} \notin \mathrm{C} \end{cases}.$$

For the monomials in the intersections of the strips,

$$(\mathrm{L} \otimes \mathrm{R/I})_{\overline{m}} = 0 \to k \to k^2 \to 0.$$

In any of the five cases above, we have that $\mathrm{H}^1(\mathrm{L} \otimes \mathrm{R/I})_{\overline{m}}$ is the first local cohomology module that is non-zero.

The interesting region is the one shaded in 2.2.1, the area in the parallelogram bounded by the translated hyperplanes bounding the ideal cones, not including the top and right edges. For an integer point in this region, we see that

$$(\mathrm{L} \otimes \mathrm{R}/\mathrm{I})_{\overline{m}} = 0 \to k \to 0 \to 0.$$

Therefore, $\mathrm{H}^0(\mathrm{L} \otimes \mathrm{R}/\mathrm{I})_{\overline{m}} \neq 0$. Since this region contains integer points, R/I is not Cohen-Macaulay.

CHAPTER 3

Cohen-Macaulay quotients of Conical Algebras

By the results of Hochster referred to earlier, we know that a conical algebra R is Cohen-Macaulay. We describe several of the methods and results used to obtain sufficient conditions that quotient rings of conical algebras R/I are again Cohen-Macaulay. We shall restrict to the case where I is generated by monomials. There are two main distinctions which characterize the results. The first is whether the conical algebra is a polynomial algebra or a more general conical algebra and the second is whether the ideal is radical or not.

In the case of polynomials rings and radical ideals, the work of Stanley and Reisner give conditions on certain simplicial complexs associated to the ideals. In the case when I is not radical, Bayer, Peeva and Sturmfels construct a simplex that computes the depth using a free resolution and the Auslander-Buchsbaum formula. When R is an arbitrary conical algebra, and if I is the ideal generated by all monomials lying off a subcomplex of faces of the cone, then I is again a radical ideal. Miller proved a necessary and sufficient condition that R/I is C-M in terms of the topology of this subcomplex.

We conclude by discussing the following open problem and some approaches to solving it: given an arbitrary conical algebra and a monomial ideal I with a minimal set generators $m_1, m_2, \ldots, m_n$, what conditions on the generators guarantee that R/I is Cohen-Macaulay?

This would combine the aspect of Miller's work on arbitrary cones, with Bayer,Sturmfels and Peeva's work on finitely generated (ie,non-radical) ideals.

3.1. Ideals in Polynomial Rings

In this entire chapter, k is an algebraically closed field.

We first consider the case when C is the positive orthant in $\mathbb{R}^n$, so that k[C] is the polynomial ring in n variables. The first case considered for polynomial rings were monomial ideals which are radical. We refer to these as Stanley-Reisner ideals. For any Stanley-Reisner ideal I, we can associate to I a subcomplex Δ of the n-simplex. Stanley and Reisner gave conditions on Δ which imply that R/I is Cohen-Macaulay. We begin by explaining these results.

3.1.1 Stanley-Reisner Ideals

Let C be the positive orthant in $\mathbb{R}^n$. A cross-section of C is the standard n-simplex. We note that $k[\mathrm{C}] = k[\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_n]$. Stanley-Reisner ideals are radical ideals generated by monomials. Stanley and Reisner gave an alternate combinatorial description of these ideals in terms of the faces of C that the generators lie off of. We introduce some definitions that will be needed for this description.

Definition 3.1.1. *A* ***simplicial complex*** Δ *on the vertex set* $V = \{v_1,v_2,\ldots,v_n\}$ *is a collection of subsets of V such that*

1)$\{v_i\} \in \Delta$ *for* $i = 1,\ldots,n$

2)$F \in \Delta$ *whenever* $F \subset G \in \Delta$

The elements of Δ are called faces and the dimension of a face F, denoted dim F, equals |F|-1. The dimension of Δ, written dim Δ is $\max_{F\in\Delta}$ {dim F}. The maximal faces under inclusion are called **facets**. Note that a facet may have dimension $<$ dim Δ.

DEFINITION 3.1.2. *A simplicial complex is said to be **pure** if all its facets have the same dimension.*

For the next definition we introduce some notation. **conv(S)** will be the convex hull of a finite set S. The interior of conv(S) relative to the vector space V will be denoted **int (conv(S),V)**.

DEFINITION 3.1.3. *Let ϕ:$V\to \mathbb{R}^n$ satisfy the following:*

1)ϕ is injective,

2)for $F \in \Delta$ of dim t, the elements of $\phi(F)$ span a t-dim subspace, denoted $<\phi(F)>$,

3)for $F \neq G \in \Delta$, the int(conv($\phi(F)$), $<\phi(F)>$) $\cap$ int(conv($\phi(G)$), $<\phi(G)>$) $= \emptyset$.

*Then $\cup_{F\in\Delta}${int(conv($\phi(F)$),$<\phi F>$)} is called the **geometric realization** of Δ.*

Let Δ be an abstract simplicial complex on the vertex set V={v_1,v_2,...,v_n}. Let $k[\Delta]$ denote the quotient ring $k[x_1, x_2, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal generated by monomials $x_{v_{i_1}} \ldots x_{v_{i_t}}$ such that $\{v_{i_1}, \ldots, v_{i_t}\} \notin \Delta$. Note that I is the ideal generated by monomials in k[C] which lie off Δ. If $k[\Delta]$ is a C-M ring, we say that Δ is a Cohen-Macaulay simplicial complex.

DEFINITION 3.1.4. *Let Δ be a simplicial complex and F a subset of the vertex set. The **star of F** is the set $st_\Delta(F) = \{G \in \Delta \mid F \cup G \in \Delta\}$ and the **link of F** is the set $lk_\Delta(F) = \{G \in \Delta \mid F \cup G \in \Delta,\ F \cap G = \emptyset\}$*

We will occasionally omit the subscript Δ in st_Δ and lk_Δ. Note that st(F) is a subcomplex of Δ and that lk(F) is a subcomplex of st(F).

The problem of showing whether a given simplicial complex is C-M becomes more tractable if one uses local cohomology. As explained in Section 2.3, the local cohomology complex can be calculated for each multidegree. Indeed, for each face F = $\{v_{i_1},\ldots,v_{i_r}\}$ of Δ, there exists a corresponding monomial $x_{v_{i_1}}x_{v_{i_2}}\ldots x_{v_{i_r}}$ and a localization, $k[\Delta]_{x_{v_{i_1}}x_{v_{i_2}}\ldots x_{v_{i_r}}}$, with respect to the multplicatively closed set $\{(x_{v_{i_1}}x_{v_{i_2}}\ldots x_{v_{i_r}})^k\}$. Reisner showed that given a multidegree $\overline{m}$, the condition that $k[\Delta]_{(x_{i_1}\ldots x_{i_r})}$ has a nonzero $\overline{m}$-graded component is equivalent to F satisfying a geometric condition in terms of Δ. Indeed, given $\overline{m} = (\overline{m}_1,\ldots,\overline{m}_n)$, define two sets

$$\overline{m}^+=\{v_i \mid \overline{m}_i > 0\} \text{ and } \overline{m}^-=\{v_i \mid \overline{m}_i < 0\}.$$

Now if $x^{\overline{m}}$ is nonzero in R_F, F $\cup\, \overline{m}^+ \in \Delta$. Also, the variables x_{v_i}, $v_i \in \overline{m}^-$, must be invertible. Thus $\overline{m}^- \subset$ F. In other words, F $\in \mathrm{lk}_{st(\overline{m}^+)}(\overline{m}^-)$. Thus, the local cohomology modules of the ring k[Δ] can be associated with the augmented chain complex of these various links. Stanley showed that the homology of the links can be computed from the topological local homology of the geometric realization of Δ. These two theorems reinterpret the C-M property of $k[\Delta]$ in terms of this geometric and topological data.

Theorem 3.1.5. (Reisner)[**14**] *Δ is C-M iff $\tilde{H}_i$(lk(F))=0 for all F $\in \Delta$ and for all $i <$ dim (lk(F)).*

Theorem 3.1.6. (Stanley)[**17**] *Let X be the geometric realization of Δ. Δ is C-M iff for all p $\in$ the interior of X and $i <$ dim X, $\tilde{H}_i$ (X,X$\setminus\{p\}$) = 0 .*

We have as a consequence of Property 1.6.3 that C-M simplicial complexes have to be pure, because every maximal chain of primes have to have the same length.

EXAMPLE 3.1.7.

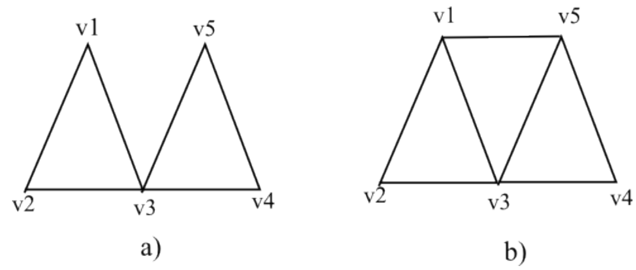

FIGURE 3.1. Stanley-Reisner ideals; a)Δ_1 and b)Δ_2

Note that in both figures above, the two-dimensional faces are elements of the respective complexes. In Fig. 3.1a), $I_{\Delta_1} = <x_1x_5, x_1x_4, x_2x_4, x_2x_5>$ and dim $k[\Delta_1] = 3$. From the figure, one can see that the link of v_3 consists of two disjoint one-dimensional faces. Thus $H_0(\text{lk } (v_3)) \neq 0$ and $k[\Delta_1]$ is not C-M. By contrast Fig. 3.1b), $I_{\Delta_2} = <x_1x_4, x_2x_4, x_2x_5 >$, but again dim $k[\Delta_2] = 3$. The edges have vertices as links, so their links vacuously satisfy the Reisner criterion. The vertices have as their links a simply connected union of edges, so their links satisfy the criterion also. Therefore, $k[\Delta_2]$ is C-M.

3.1.2 Non-Radical Ideals in Polynomial Rings

We now turn to the case where the ideals in the polynomial ring are not radical. We can no longer simply analyze the faces of the cone as in the radical case. The location of the ideal generators in relation to the faces of C and to each other play a crucial role. We will exploit the fact that we have a multigrading on the monomials.

We shall explain the results of Bayer, Peeva, and Sturmfels [**1**]. The class of non-radical ideals they consider are so-called *generic* ideals. They construct the least common multiple (lcm) lattice of the minimal monomial generators. From this lattice is constructed two simplicial complexes, the so-called Taylor complex and Scarf complex, that encodes the locations of the ideal generators relative to the faces of C. Among other results, they deduce the global projective dimension of R/I from this polyhedron. Then using the Auslander-Buchsbaum formula, they get sufficient conditions for R/I to be Cohen-Macaulay.

3.1.2.1. *The Taylor Complex and Taylor Resolution.* We again view R, the polynomial ring in n variables as a conical algebra k[C] where C is the positive orthant in $\mathbb{R}^n$. Let I $= <\mathrm{x}^{m_1}, \mathrm{x}^{m_2}, \ldots, \mathrm{x}^{m_d}> \subset$ R be a monomial ideal. Consider Δ, the full d-1 simplex. We label each vertex with a distinct ideal generator and each face of Δ with the lcm of its vertices. For a face K $\in \Delta$, let m_K be the label for K. We define $\mathrm{m}_\emptyset = 1$. This labeled simplex is the **Taylor complex**. Let a_K be the exponent vector for m_K. Let R(-a_K) be the graded free R-module with one generator e_K in multidegree a_K. The **Taylor resolution** of R/I is

$$0 \rightarrow \bigoplus_{\dim \mathrm{K} = \mathrm{d}-1} \mathrm{R}(-\mathrm{a}_K) \rightarrow \bigoplus_{\dim \mathrm{K} = \mathrm{d}-2} \mathrm{R}(-\mathrm{a}_K) \rightarrow \ldots \rightarrow \mathrm{R} \rightarrow \mathrm{R/I} \rightarrow 0.$$

The differential is defined componentwise

$$\partial\colon \mathrm{R}(-\mathrm{a}_K) \rightarrow \mathrm{R}(-\mathrm{a}_{K'})$$

to be

$$\partial(\mathrm{e}_K) = \begin{cases} (-1)^k \frac{m_K}{m_{K'}} e_{K'}, & \text{if } K' = K \setminus k \\ 0, & \text{otherwise} \end{cases},$$

Note that $\frac{m_K}{m_{K'}}$ is still a monomial since the lcm of a subset of monomials must divide the lcm of the whole set. Note that the Taylor resolution is just the chain complex of the Taylor complex with a homogenized (multidegree-preserving) differential.

THEOREM 3.1.8. (Taylor)[**18**] *The Taylor Complex is a free resolution for R/I.*

EXAMPLE 3.1.9.

Let I = $< x^4y^6z^7, x^5y^7z^3, x^6y^7z^2 >$. The Taylor resolution is

$$0 \to R \to R^3 \to R^3 \to R \to R/I.$$

Note that the power of R in each homological degree i is the number of i-faces of the Taylor Complex. The associated Taylor complex is

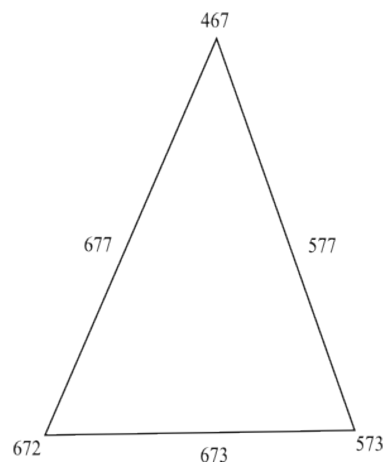

FIGURE 3.2. The Taylor complex of I = $< x^4y^6z^7, x^5y^7z^3, x^6y^7z^2 >$, with faces labeled

3.1.2.2. *The Scarf Complex and Scarf Resolution.* The Taylor resolution is far from minimal, because in general there will be distinct subsets K,L which give the same lcm. We know that any free resolution is a direct sum of the minimal resolution with trivial algebraic complexes. In [**1**], a genericity condition is found that simplifies the homological behavior.

DEFINITION 3.1.10. *I is a* ***generic*** *monomial ideal if* $I = <x^{m_1}, x^{m_2}, \ldots, x^{m_t}>$ *and no variable has the same nonzero exponent for any two generators.*

To pick out the factors of the minimal resolution we will consider the **Scarf Complex** of I:

$$\Delta_I := \{K \subset \{1,\ldots,d\} \mid m_K \neq m_J \text{ for any } J \subset \{i,\ldots,d\},\ J \neq K\}.$$

Example 3.1.11.

This is the Scarf complex of I from Example 3.1.9.

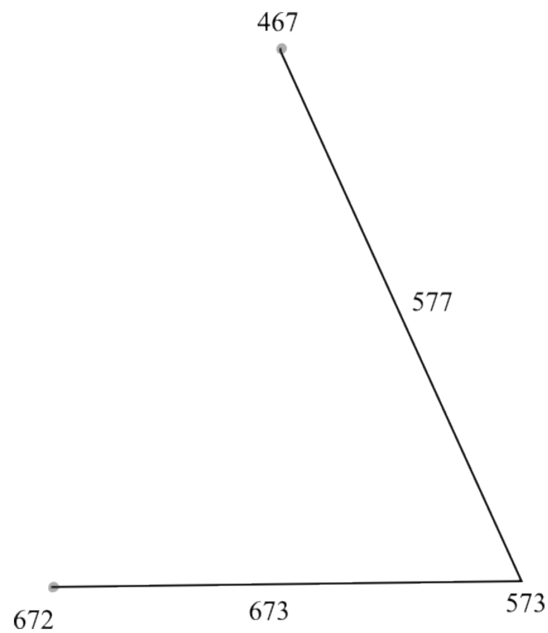

Figure 3.3. The Scarf complex of I = $< x^4y^6z^7, x^5y^7z^3, x^6y^7z^2 >$, with faces labeled

Note that this complex defines a subresolution, $\mathbb{F}_{\Delta_I}$, of the Taylor resolution closed under ∂. We call this **the Scarf resolution** of R/I. The importance of the Scarf complex and the induced resolution is shown in the following theorem [**1**].

Theorem 3.1.12. *If I is generic, $\mathbb{F}_{\Delta_I}$, the resolution defined by the Scarf Complex Δ_I, is a minimal free resolution of R/I.*

If I is generic, then this theorem allows the depth of R/I to be calculated using the Auslander-Buchsbaum formula to calculate the depth. However, Bayer, Peeva and Sturmfels prove a stronger result. They find an irreducible primary decomposition for I and calculate the depth of R/I from the depths of these associated primes.

If we enlarge the set of generators to include monomials, x_i^D, on the coordinate axes for sufficiently large D, this forms a new ideal $I^\star$, which has t + n generators. For every facet F of $\Delta_{I^\star}$, there is an irreducible ideal I_F. By the following theorem, one can construct the irreducible decomposition of I.

THEOREM 3.1.13. *A generic ideal I is the intersection of the irreducible ideals I_F, where F ranges through all facets of $\Delta_{I^\star}$. This intersection is minimal.*

One can now calculate the depth of I by using this minimal irreducible decomposition. By the Auslander-Buchsbaum formula, the depth R/I = depth R - pd R/I. The depth of R is n. The projective dimension of R/I is the maximum of the dimensions of the facets of Δ_I, which, in general, is not a pure complex. However, every facet of Δ_I extends to a facet of $\Delta_{I^\star}$, which is pure of dim n-1. Conversely every facet of $\Delta_{I^\star}$ contains a face of Δ_I. Thus, determining whether R/I is Cohen-Macaulay is equivalent to determining the dimensions of these irreducible components.

THEOREM 3.1.14. *Let I be a generic ideal of R, the polynomial ring in n variables. Then depth of R/I =* $\min_{\textit{facets } F \textit{ of } \Delta_{I^\star}} \{\textit{dim } I_F\}$.

PROOF. We have the following string of equalities.

$$\begin{aligned} \text{depth R/I} &= \min_{\text{facets G of } \Delta_I} \{\text{n - dim G}\} \\ &= \min_{\text{facets F of } \Delta_{I^\star}} \{\text{n - } |\ \text{F} \cap \{1,\ldots,\text{t}\}|\ \} \\ &= \min_{\text{facets F of } \Delta_{I^\star}} \{|\ \text{F} \cap \{t+1,\ldots,t+n\}\ |\} \end{aligned}$$

$$= \min_{\text{facets F of } \Delta_{I^\star}} \{\dim \mathrm{I}_F\}.$$

□

Example 3.1.15.

Let R=k[x,y,z]. Let I =<x^2yz^4, x^4y^7z>. $\Delta_{I^\star}$ is composed of six triangles which correspond to the six components in the decomposition of I. I $= < x^2 > \cap < y > \cap < z > \cap < x^4, y^7 > \cap < x^4, z^4 > \cap < y^7, z^4 >$. The depth of R/I = 1, and R/I is not C-M.

Example 3.1.16.

Again let R=k[x,y,z]. Let I = < x^2z, y^{10} >. In this case, $\Delta_{I^\star}$ is made up of two triangles, which share an edge. The irreducible decomposition of I is given by < x^2 , y^{10} > ∩ <y^{10}, z>. The depth of R/I is 1 and R/I is Cohen-Macaulay.

3.2. Radical Ideals in General Conical Algebras

If we move from a polynomial ring k[C] where C is the positive orthant to a more general positive rational cone C, there are natural generalizations of the Stanley-Reisner ideals. In this case, the positive dimensional faces of the cone form a complex Γ which plays the role of the n-simplex. We will let I_Δ be the ideal generated by monomials that lie off a given subcomplex $\Delta \subset \Gamma$. These ideals are radical, just as in the polynomial case. Miller generalizes the definition of a link to apply to general polyhedral complexes, such as the face complex of C. He then defines a new algebraic complex to calculate the cohomology of the links. All the definitions and results in the following can be found in [**13**].

3.2.0.3. *Cohen-Macaulay Complexes.* First, we generalize the notion of Δ being Cohen-Macaulay.

For a face G, recall that <G> is the linear span of G. Consider a hyperplane H of dim = codim <G>. Let H intersect C such that every face containing G intersects H and H $\cap$ G = $\emptyset$. Define lk(G) = H $\cap$ C. Note that a k-dimensional face F containing G is a k-dim G dimensional face of lk(G). For completeness, we include the empty face in lk(G). Now, let Δ_G =$\{$F $\in \Delta \mid$ F $\supset$ G$\}$. The cochain complex $\mathcal{C}^{\bullet}(\Delta_G)$ is isomorphic to the reduced cochain complex $\tilde{\mathcal{C}}^{\bullet}$(lk G), with the shifting where the empty face is in homological degree dim G.

Definition 3.2.1. *$H^i(\mathcal{C}^{\bullet}(\Delta_G))$ is called the* ***local cohomology $H^i_G(\Delta)$ of Δ near G****.*

Definition 3.2.2. *The polyhedral complex Δ is a* ***Cohen-Macaulay complex*** *if the local cohomology near every face $G \in \Delta$ satisfies $H^i_G(\Delta) = 0$ for $i <$ dim Δ.*

Theorem 3.2.3. (Miller)[**13**] *$k[C]/I_\Delta$ is a C-M ring iff Δ is Cohen-Macaulay.*

To verify that R = k[C]/I_Δ is Cohen-Macaulay, Miller constructs a double complex that is an irreducible resolution for R, using an idea that appeared previously in a paper by Danilov [**4**]. This resolution is far from minimal, but if R is C-M, a cancellation occurs that causes the vertical differential to be a minimal resolution. Miller shows that R is Cohen-Macaulay iff this double complex gives a minimal resolution of R/I. We discuss this double complex and its consequences.

3.2.0.4. *Irreducible Resolutions and the Zeeman Double Complex.*

Definition 3.2.4. *An **irreducible resolution** of a k[C]-module M is an exact sequence*

$$0 \to M \to W^0 \to W^1 \to \ldots \text{ where } W^t = \bigoplus_{j=1}^{\mu_t} k[C]/I^{tj}$$

*such that each I^{ij} is an irreducible ideal. The resolution is **minimal** if all the μ_t are minimized.*

For a face F of C, let kF be the 1-dimensional vector space spanned by F in $\mathbb{Z}^n$-graded degree $\mathbf{0}$. For each face F $\in \Delta$, let k[F] be the conical algebra for F, viewed as a quotient of k[C] and denote e$_F$ as the generator of k[F] in degree $\mathbf{0}$. Consider the k[C]-module D(Δ) = $\oplus_{F \supseteq G}$ kF $\otimes$ k[G] generated by

$$\{\mathrm{F} \otimes \mathrm{e}_G \mid \mathrm{F}, \mathrm{G} \in \Delta \text{ and } \mathrm{F} \supseteq \mathrm{G}\},$$

with k[C] acting on the right factor. Define the Zeeman double complex D(Δ) such that D(Δ)$_{pq}$ is generated by

$$\{\mathrm{F} \otimes \mathrm{e}_G \mid \text{p=dim F and -q=dim G}\}$$

Define the vertical differential ∂ and the horizontal differential δ as

$$\partial(\mathrm{e}_G) = \sum_{G', \text{maximal in } G} \epsilon(\mathrm{G},\mathrm{G}')\mathrm{e}_{G'} \text{ and } \delta(\mathrm{F}) = \sum_{F, \text{maximal in } F'} (-1)^q \epsilon(\mathrm{F}',\mathrm{F})\mathrm{F}',$$

where ϵ is some incidence function on Δ. This gives us the diagram:

$$\begin{array}{ccc} F \otimes \partial e_G & & \\ \uparrow & & \\ F \otimes e_G & \longrightarrow & \delta F \otimes e_G \end{array} .$$

The importance of this double complex is given by the following result.

Theorem 3.2.5. *The total complex of D(Δ) is an irreducible resolution of R.*

This complex is an analog to the Taylor complex, in the sense that they both provide natural resolutions of the quotient rings under consideration.

Miller shows that for a fixed G, the horizontal cohomology gives the local cohomology of Δ near G. The total complex of the Zeeman double complex is, in general, a non-minimal irreducible resolution of R.

Definition 3.2.6. *The $\mathbb{Z}^n$-graded Zeeman spectral sequence for D(Δ) is the spectral sequence $\mathbb{Z}E^{\bullet}_{pq}$ obtained by taking horizontal homology first.*

The following theorem gives a characterization of the Cohen-Macaulay property in terms of irreducible resolutions coming from D(Δ).

Theorem 3.2.7. (Miller) Δ *is Cohen-Macaulay iff $\mathbb{Z}E^{\bullet}_{pq}$ is a minimal linear irreducible resolution for R.*

3.3. Extending Results to Non-Radical Ideals in Conical Algebras

Suppose we now consider a non-radical monomial ideal in a conical algebra. One would like to to state conditions on a finitely generated monomial ideal I so that k[C]/I is Cohen-Macaulay. The methods for the radical ideals are not directly applicable. We would like to develop a theory in the non-radical ideal case so that the Stanley-Reisner class of ideals could be considered as special cases. The first natural technique would be to extend the method of Sturmfels, Bayer and Peeva. However, there is a fundamental problem for a general conical algebra, because there is not a well-defined notion of a least common multiple. Indeed, k[C] is a quotient of the polynomial ring, so there will be relations between the monomials. However, we can extend the definition of a generic ideal, so that no generator lies on the same hyperplane, unless the hyperplane supports

C.

An approach we are presently considering is the local cohomology complex $L^\bullet$ which makes use of the geometry, i.e., the face lattice, of the cone. The $L^\bullet$ complex (Section 2.2) allows us to exploit the multigrading of $k[C]$. The goal would be to again calculate the local relative cohomology of some topological space, like Stanley and Reisner. We cannot directly use the Stanley-Reisner method because in general the rings will have non-zero monomials lying in the interior of C. We seek to partition the vector space so that all the monomials in each part have the same graded $L^\bullet$ complex. This method was used by Hochster to show that the conical algebras themselves are Cohen-Macaulay. Then, one could check if there exists any multidegrees such that the local cohomology is non-zero below the dimension or R/I. In doing so, the results for the radical ideals could be seen as a special case.

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